Malaysian Journal of Mathematical Sciences 20(2): 641–657 (2026)
https://doi.org/10.47836/mjms.20.2.10

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Multigrid and Bi-Conjugate Gradient Stabilized Methods for the Ill-Posed Complex Helmholtz Equation: Comparative Studies

Sweilam, N. H. * [1], Ibrahim, M. H. [2], Ramadan, M. [2,3], and Eseily, A. [2]

[1]*Department of Mathematics, Faculty of Science, Cairo University, Cairo, Egypt*
[2]*Department of Mathematics, Faculty of Science, Port Said University, Port Said, Egypt*
[3]*Department of Mathematics, College of Sciences, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia*

E-mail: nsweilam@sci.cu.edu.eg
**Corresponding author*

Received: 27 May 2025
Accepted: 15 October 2025

Abstract

In this study, we develop and analyze efficient numerical schemes for solving the ill-posed complex Helmholtz equation in both two and three spatial dimensions under Dirichlet boundary conditions. The spatial discretization is carried out using the classical finite difference method and the nonstandard finite difference method, with the latter offering enhanced stability. The resulting linear systems are large, sparse, and highly sensitive to perturbations, reflecting the ill-posed nature of the problem. To address these challenges, two advanced iterative solvers are employed: the multigrid method, with the generalized minimal residual used as a smoother, and the bi-conjugate gradient stabilized method. Numerical experiments compare the computed solutions with solutions to verify accuracy, and condition numbers are evaluated to assess ill-posedness and stability. The results demonstrate that multigrid achieves robust and stable convergence, particularly in cases where the bi-conjugate gradient stabilized method fails, while the nonstandard finite difference method consistently improves numerical stability compared with the classical finite difference method.

Keywords: complex Helmholtz equation; nonstandard finite difference method; finite difference method; iterative solvers; ill-posed systems of equations.

1 Introduction

The complex Helmholtz equation is a key tool for describing time-harmonic wave propagation in various fields such as acoustics, and electromagnetics. Despite its widespread use, accurately solving this equation especially at high frequencies poses considerable numerical difficulties. This is largely due to the equation's indefinite, complex-valued, and non-Hermitian characteristics, which make traditional computational methods less effective. To address these numerical challenges, recent research has introduced advanced strategies, including learning–based operator approaches and meshless Galerkin least-square methods [7, 22]. In addition, a wide range of numerical strategies has been developed over the years to enhance accuracy and efficiency, such as high-order finite element methods [8], least-squares finite element methods [35], spectral element methods [27], radial basis function-generated finite difference schemes [25], and hybridizable discontinuous Galerkin methods [6].

Many numerical methods have useful features, but they can be hard to implement and often need a lot of computing power especially for problems of higher dimensions. On the other hand, the finite difference method (FDM) is still widely used because it is simple and flexible. It works by using values on a grid to approximate derivatives, which makes it popular in physics and engineering. However, FDM has some drawbacks. It can struggle with accuracy and stability, and it may not capture important features of the solution. These problems become more noticeable in complex situations or when the results are very sensitive to changes in parameters [24, 34]. An efficient algorithm based on the FDM with successive over-relaxation was developed for solving the Poisson–Boltzmann equation, resulting in a significant reduction in computational time. The method included an improved relaxation parameter, highlighting the importance of enhancing iterative strategies to accelerate the numerical solutions of elliptic equations [28].

To overcome these challenges, nonstandard finite difference (NSFD) schemes have been introduced as a more reliable alternative, providing accurate and stable numerical solutions for dynamical systems [2]. NSFD approaches have been successfully applied to various problems, including fractional Navier–Stokes equations [33] and multi–dimensional second-order systems in non-smooth mechanics [11], demonstrating improved stability and adaptability. Recent work has further confirmed the efficiency of NSFD-based schemes when combined with advanced iterative solvers such as bi-conjugate gradient stabilized (BiCGSTAB) and multigrid (MG), achieving faster convergence and higher accuracy, particularly on fine grids [17].

The Helmholtz equation at high wave numbers has attracted significant attention from researchers due to the major challenges in maintaining numerical accuracy and stability, as well as in correctly capturing wave properties such as phase and amplitude [21, 23]. Unlike the positive definite Helmholtz equation, the indefinite Helmholtz equation is much harder to solve using classical iterative methods. Standard Krylov solvers and common preconditioners such as MG often fail or converge very slowly, especially for large wave numbers. This difficulty has motivated the development of specialized techniques to improve stability and convergence for indefinite Helmholtz problems [16].

Linear systems of equations that arise from the discretization of the Helmholtz equation present considerable difficulties. Due to their large size, sparse structure, and high computational cost, direct solution methods are often impractical. This study utilizes advanced iterative solvers known for their scalability and efficiency in handling large-scale problems [14, 3]. One popular approach is the BiCGSTAB method, which is specifically developed to handle non-symmetric and indefinite linear systems. It is particularly appreciated for its ability to stabilize the iteration process, leading to better convergence when dealing with complex-valued problems [38]. MG methods are also

used because they work with multiple levels of spatial grids to quickly reduce errors at different scales. However, how well they perform depends a lot on the smoothing techniques applied [40, 18]. In recent studies, MG has been effectively combined with high-order compact difference schemes, such as the sixth-order approach, leading to improved accuracy and faster convergence compared with conventional lower-order methods [19]. In this study, the generalized minimal residual (GMRES) method is used as a smoother within the MG cycles to help speed up convergence by effectively reducing high-frequency errors [13, 15]. The analysis focuses on important factors like how fast the method converges, the accuracy of the solution, and the overall reliability of the solver.

Recent advances in integral-equation methods and numerical analysis have strengthened the theoretical and computational toolkit available for nonlocal and fractional problems. In particular, Mahdy [26] studied stability, existence, and uniqueness results for a fractional model of glioblastoma multiforme using the Caputo–Fabrizio derivative and discussed the numerical integration steps required for that model, highlighting the importance of rigorous stability and uniqueness analysis when designing numerical schemes for nonlocal operators.

In summary, the contributions of this study can be stated as follows. First, we present a comparative investigation of two iterative solvers, the MG method and the BiCGSTAB method, for the ill-posed complex Helmholtz equation in both 2D and 3D cases. Second, we employ both the classical FDM and the NSFDM, where the latter demonstrates improved stability near the boundaries. Finally, a novel aspect of this work is the use of GMRES as a smoother within the MG framework, which enhances convergence for indefinite and oscillatory systems. These contributions underline the originality of the proposed framework and justify the motivation of this study.

2 FDM and NSFD Methods

The Helmholtz equation describes time-harmonic wave propagation and is typically written as:

$$\nabla^2 u(x,y,z) + k^2 u(x,y,z) = 0, \qquad (x,y,z) \in [0,1] \times [0,1] \times [0,1], \tag{1}$$

where k is the wavenumber, and ∇^2 denotes the Laplacian operator in three dimensions:

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2}$$

We impose Dirichlet boundary conditions on all boundaries of the domain:

$$\begin{cases} u(0,y,z) = h_1(y,z), & (y,z) \in [0,1] \times [0,1], \\ u(1,y,z) = h_2(y,z), & (y,z) \in [0,1] \times [0,1], \\ u(x,0,z) = h_3(x,z), & (x,z) \in [0,1] \times [0,1], \\ u(x,1,z) = h_4(x,z), & (x,z) \in [0,1] \times [0,1], \\ u(x,y,0) = h_5(x,y), & (x,y) \in [0,1] \times [0,1], \\ u(x,y,1) = h_6(x,y), & (x,y) \in [0,1] \times [0,1]. \end{cases} \tag{3}$$

Here, the functions $h_1, h_2, \ldots, h_6$ are the given boundary values, and the solution $u(x,y,z)$ is sought in the domain $[0,1] \times [0,1] \times [0,1]$.

At an interior grid point (i, j, k), the Helmholtz equation is discretized using a second-order finite difference (FD) scheme, resulting in the following form:

$$\begin{aligned}\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{(\Delta x)^2} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{(\Delta y)^2} \\ + \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{(\Delta z)^2} + k^2 u_{i,j,k} = 0.\end{aligned} \tag{4}$$

On a uniform grid where the mesh sizes in all directions are equal, $\Delta x = \Delta y = \Delta z = h$, the equation can be rearranged as:

$$u_{i,j,k} = \frac{-1}{-6 + h^2 k^2} \left(-u_{i+1,j,k} - u_{i-1,j,k} - u_{i,j+1,k} - u_{i,j-1,k} - u_{i,j,k+1} - u_{i,j,k-1}\right). \tag{5}$$

Equation (4) can also be discretized using the Nonstandard Finite Difference (NSFD) framework. At an interior grid point (i, j, k), the NSFD discretization of the Helmholtz equation takes the form:

$$\begin{aligned}\frac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{\phi_1(\Delta x)} + \frac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{\phi_2(\Delta y)} \\ + \frac{u_{i,j,k+1} - 2u_{i,j,k} + u_{i,j,k-1}}{\phi_3(\Delta z)} + k^2 u_{i,j,k} = 0.\end{aligned} \tag{6}$$

Here, $u_{i,j,k}$ represents the numerical approximation at the grid point (x_i, y_j, z_k), while $\phi_1(\Delta x)$, $\phi_2(\Delta y)$, and $\phi_3(\Delta z)$ are the nonstandard denominator functions. Unlike the standard FD scheme where denominators are simply $(\Delta x)^2$, $(\Delta y)^2$, and $(\Delta z)^2$, these functions are carefully constructed to better capture the oscillatory nature of wave propagation.

Following the NSFD design principles, the denominator functions are defined as:

$$\phi_1(\Delta x) = \frac{h}{\sqrt{1 + (M_1 h)^2/12}}, \quad \phi_2(\Delta y) = \frac{h}{\sqrt{1 + (M_2 h)^2/12}}, \quad \phi_3(\Delta z) = \frac{h}{\sqrt{1 + (M_3 h)^2/12}},$$

where h denotes the grid spacing (assumed uniform for simplicity), and M_1, M_2, M_3 are constants related to the wavenumber k and the propagation direction of the solution.

Remarks.

- These nonstandard denominator functions act as correction factors that compensate for phase errors in the numerical dispersion of standard FD schemes.
- By incorporating the terms M_1, M_2, and M_3, the NSFD scheme aligns more closely with the analytical properties of the solution, particularly for oscillatory problems like the Helmholtz equation.
- This adjustment improves stability, reduces numerical dispersion, and allows accurate approximation with coarser grids compared to classical FDM.

After using both the FDM and NSFD to discretize the complex Helmholtz equation schemes, the continuous PDE is converted into linear algebraic system. This step is important because it lets us use efficient iterative methods to find an approximate solution. The effectiveness of the

NSFD approach in improving numerical stability and preserving essential properties of the original problem has also been demonstrated in other studies, such as Raza et al. [31], who successfully applied the NSFD scheme to simulate a fractional-order epidemiological model with enhanced stability and accuracy.

In this study, we focus on two advanced iterative methods to tackle these difficulties. The first is the MG method, known for its fast and efficient error reduction at different levels, helping speed up convergence even with oscillating problems. The second is the BiCGSTAB algorithm, which works well for solving large, sparse, and non-symmetric systems that often come from Helmholtz equation discretizations.

3 Iterative Methods

Recent studies have focused on refining classical iterative schemes to enhance convergence and stability, particularly for stiff or ill-conditioned problems. For instance, Xu et al. [42] proposed a refined Successive Over-Relaxation (SOR) method based on a linear rational finite difference formulation, demonstrating faster convergence and improved numerical accuracy for integro-differential equations. Such developments highlight the continuous effort to optimize iterative solvers for complex numerical models.

Iterative methods are basic techniques used to solve linear systems like,

$$Ax = b, \tag{7}$$

where

$$\begin{aligned} A \in \mathbb{R}^{n\times n} &\text{ is a non-singular square matrix,}\\ b \in \mathbb{R}^{n} &\text{ is a right-hand side vector,}\\ x \in \mathbb{R}^{n} &\text{ is the unknown solution vector.} \end{aligned}$$

These methods are effective when systems are large or sparse, where the computational cost of direct methods, such as Gaussian elimination can be too expensive. Iterative solvers work by repeatedly improving an initial guess through simple calculations until a set accuracy or stopping condition is reached [41]. Iterative methods are often faster and more efficient than direct solvers in real-world applications like numerical simulations, structural analysis, and solving partial differential equations, due to their reduced memory requirements and computational overhead.

Prominent iterative algorithms studied in the literature include the Jacobi method, Gauss-Seidel method, gradient-based approaches, as well as more advanced techniques like the conjugate gradient method and BiCGSTAB, which are widely used for solving symmetric and non-symmetric linear systems, respectively. In addition, MG methods have gained significant attention due to their optimal complexity and their ability to accelerate convergence through multi-level correction strategies.

3.1 MG method

MG algorithms solve this problem by utilizing a series of grids with different resolutions. The main components of a MG method are [20]:

1. **Smoothers (e.g., Gauss–Seidel, Jacobi):**
   Applied on all levels except the coarsest to damp high-frequency errors.
2. **Coarse-grid solver:**
   Provides an exact or approximate solution on the coarsest level, where the system size is sufficiently small.
3. **Inter-grid transfer operators:**
   - *Restriction* transfers residuals from fine to coarse grids.
   - *Prolongation* interpolates corrections back from coarse to fine grids.
4. **Cycling strategy:**
   Controls how the method transitions between different grid levels (e.g., V-cycle, W-cycle).

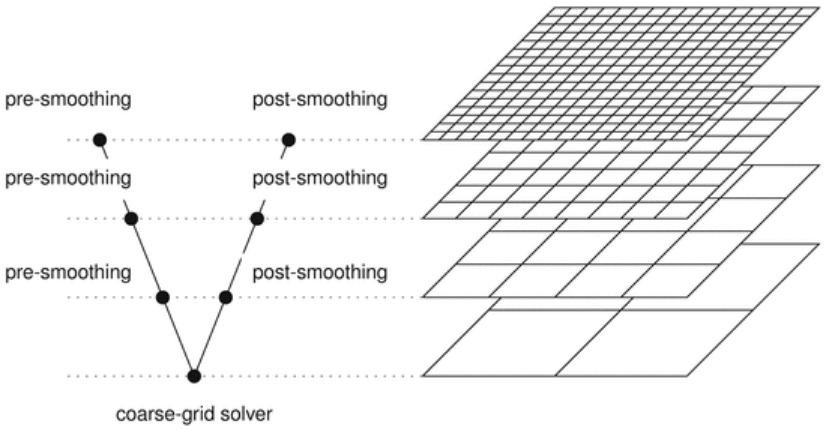


Figure 1: The MG (V-cycle) process, including pre-smoothing, coarse-grid correction, and post-smoothing at various grid levels.

On uniform fine grids, when we use classical iterative methods like Jacobi or Gauss-Seidel may require a large number of iterations, especially for problems such as the Poisson equation. This is because they are inefficient at removing low-frequency errors. MG methods overcome this by transferring the error from fine grid to coarser grids thus accelerating overall convergence. For an in-depth discussion of MG methods, the reader is referred to standard references [37, 4]. A particularly important component of the multigrid method is the smoother. Classical smoothers are effective for symmetric positive-definite systems, but may perform poorly for non-symmetric or indefinite problems. To handle such cases, GMRES can be used as a smoother. GMRES is well-suited for non-symmetric, ill-conditioned linear systems and provides a more robust smoothing strategy in those contexts. This work investigates the use of GMRES as a smoother within a MG framework for solving the linear system defined in (7).

In Algebraic Multigrid (AMG), the multilevel structure is constructed based on the matrix rather than the physical grid. The goal is to eliminate the smooth error, which standard relaxation fails to reduce, through coarse-grid correction [30]. This process involves solving the residual equation,

$$Ae = r, \tag{8}$$

on a coarser grid, where e is the error and r is the residual. The computed error is then interpolated back to the fine grid, and the current solution is updated as,

$$u \leftarrow u + e. \tag{9}$$

For notational clarity, the matrix $A = [a_{ij}]$ is indexed so that i and j correspond to nodes on a computational grid Γ, and the vector $x = [x_i]$ represents the solution at each grid point.

3.1.1 Core components and structure of the AMG method

The AMG method relies on several key components that are constructed during a preliminary setup phase. These components are:

1. A sequence of M grid levels represented by index sets of computational nodes,
$$\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_M.$$
2. A series of system matrices $A_1, A_2, \ldots, A_M$, each corresponding to a different grid level.
3. Interpolation operators $P_1, P_2, \ldots, P_{M-1}$ that transfer corrections from coarse to fine levels.
4. Restriction operators $R_1, R_2, \ldots, R_{M-1}$ that transfer residuals from fine to coarse levels.
5. Smoothers $S_1, S_2, \ldots, S_{M-1}$, used to damp high-frequency error components.

These elements are defined during the setup phase, which proceeds as follows:

Algorithm 1 AMG setup phase

Initialize the fine-level grid:
    Set: $\Gamma_1 = \Gamma$
for $k = 1$ to $M - 1$ **do**
    Partition the set Γ_k into two disjoint subsets: the set of coarse points C_k and the set of fine points F_k. The choice of coarse points is crucial:
    Coarse points C_k form the next grid level Γ_{k+1}.
    Fine points F_k are interpolated from C_k.
    Selection can be based on:
        Ruge–Stüben coarsening (strength of connections).
        Aggregation (grouping nodes into aggregates).
    The objective is to capture smooth error components on coarse grids while ensuring that the coarse problem remains significantly smaller.
    Define the next coarser level: $\Gamma_{k+1} = C_k$.
    Construct the interpolation operator P_k.
    Define the restriction operator R_k (commonly $R_k = P_k^T$).
    Form the coarse-level system matrix:
        $A_{k+1} = R_k A_k P_k$.
    Prepare the smoother S_k.
end for

After the setup is complete, AMG proceeds with a recursive MG cycle, most often a V-cycle denoted as MGV, following the notation in [9, 1]. While other cycles (e.g., W-cycle or F-cycle) exist, this work focuses on the V-cycle approach.

Algorithm 2 Multigrid V-Cycle (MGV)

if $k = M$ (coarsest level) **then**
  Solve the system $A_M x_M = b_M$ directly using a direct solver.
else
  Apply the smoother S_k for μ_1 iterations to approximately solve $A_k x_k = b_k$.
  Compute the residual: $r_k = b_k - A_k x_k$ (preparing for coarse-grid correction).
  Restrict the residual to the coarser grid: $r_{k+1} = R_k r_k$.
  Recursively call the V-cycle on the next coarser level:
    MGV$(A_{k+1}, R_{k+1}, P_{k+1}, S_{k+1}, e_{k+1}, r_{k+1})$.
  Interpolate the error back to the fine grid: $e_k = P_k e_{k+1}$.
  Correct the solution: $u_k = u_k + e_k$.
  Apply the smoother S_k again for μ_2 iterations to further reduce the error.
end if

AMG methods have gained renewed attention due to their effectiveness in solving large-scale problems on unstructured grids. Recent studies have explored their scalability and applicability across various physical models, highlighting both their strengths and limitations. Moreover, current research trends focus on improving the robustness of AMG in cases where standard formulations perform poorly [29]. In classical AMG, the smoothing step is typically performed using simple stationary iterative methods, such as Jacobi, Gauss–Seidel, or an incomplete LU (ILU) factorization as in [5]. These smoothers are designed to effectively reduce high-frequency error components with minimal computational cost. However, in this work, we adopt a non-standard approach by employing GMRES as the smoother at each level. This decision is motivated by the fact that traditional smoothers often perform poorly for the type of systems considered. By applying GMRES with a small, fixed number of iterations (typically two or three), we achieve a more robust smoothing effect while keeping the computational overhead reasonable.

3.2 GMRES method

The GMRES method is one of the most widely used Krylov subspace iterative algorithms for solving large, sparse, and nonsymmetric linear systems. Unlike direct methods, which may become computationally expensive or infeasible for high-dimensional problems, GMRES provides an efficient iterative alternative by minimizing the residual norm over a sequence of Krylov subspaces. Since its introduction over 35 years ago [43], GMRES has undergone several developments and optimizations. The core idea behind GMRES lies in projecting the original system onto a lower-dimensional subspace and then, solving the resulting least-squares problem to find the best approximate solution within that subspace. This approach not only accelerates convergence in many practical problems but also offers flexibility in terms of the restart technique. To address stability concerns in certain scenarios, Reichel and Ye proposed a breakdown-free variant of GMRES designed particularly for singular systems [32], and improved methods for solving large-scale ill-conditioned systems [12], have enhanced its applicability for large and difficult problems.

3.3 The BiCGSTAB method

The BiCGSTAB method is an effective iterative technique for solving large, sparse, and nonsymmetric linear systems. It belongs to the broader family of Krylov subspace methods, which are widely recognized for their ability to efficiently tackle large-scale problems, particularly when

direct solvers are computationally prohibitive. The BiCGSTAB method originates from the Conjugate Gradient Squared (CGS) algorithm developed by Sonneveld [36], which itself is based on the Bi-Conjugate Gradient (BiCG) method. The BiCG approach uses the Lanczos biorthogonalization process [39]. These methods combine to form a strong approach for solving non-symmetric linear systems iteratively, which are frequently encountered in science and engineering problems.

4 Numerical Tests

To evaluate the effectiveness of the proposed method, we examine two problems governed by the complex Helmholtz equation. These problems are discretized using both the NSFD and the Standard Finite Difference Method (SFDM). The resulting linear systems are solved with two iterative solvers: MG-GMRES and BiCGSTAB. The analysis is carried out for multiple values of the wave number k to assess the method's robustness under varying conditions. We present visual results of the computed solutions along with their error distributions to demonstrate the accuracy of the approximations. A consistent convergence criterion of $\varepsilon = 10^{-15}$ is applied to all grid sizes N also compute condition number (COND).

The accuracy of the computed solutions is evaluated through error norms in both the real and imaginary parts:

- $L_2^{\text{Real}}, L_2^{\text{Imag}}$ and combined L_2 norm,
- $L_\infty^{\text{Real}}, L_\infty^{\text{Imag}}$ and combined L_∞ norm.

$$\begin{cases} L_2^{\text{Real}} &= \left[\sum_{i=0}^{N}\sum_{j=0}^{N} |u(x,y,0.5) - \tilde{u}(x,y,0.5)|^2\right]^{1/2}, \\ L_2^{\text{Imag}} &= \left[\sum_{i=0}^{N}\sum_{j=0}^{N} |v(x,y,0.5) - \tilde{v}(x,y,0.5)|^2\right]^{1/2}, \\ |L_2| &= \sqrt{\left(L_2^{\text{Real}}\right)^2 + \left(L_2^{\text{Imag}}\right)^2}. \end{cases} \tag{10}$$

$$\begin{cases} L_\infty^{\text{Real}} &= \max_i \max_j |u(x,y,0.5) - \tilde{u}(x,y,0.5)|, \\ L_\infty^{\text{Imag}} &= \max_i \max_j |v(x,y,0.5) - \tilde{v}(x,y,0.5)|, \\ |L_\infty| &= \sqrt{\left(L_\infty^{\text{Real}}\right)^2 + \left(L_\infty^{\text{Imag}}\right)^2}, \end{cases} \tag{11}$$

where $\tilde{u}$ and $\tilde{v}$ are the approximations of u and v, respectively.

The Helmholtz equation, which arises in various wave propagation and scattering applications, is particularly challenging to solve numerically due to its indefinite and oscillatory nature. Cools et al. [10] proposed a new level-dependent coarse grid correction scheme for indefinite Helmholtz problems, which demonstrated improved MG performance and motivates the present study of iterative solvers for such challenging systems.

Example 4.1. *Consider the following two-dimensional Helmholtz equation,*

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + K^2 U = 0, \tag{12}$$

and the exact solution is given by,

$$U(x, y) = e^{i(k_1 x + k_2 y)}, \tag{13}$$

where

$$k_1 = k\cos\theta, \quad k_2 = k\sin\theta.$$

Tables 1–3 present the numerical results obtained using the MG-GMRES method, the NSFD method and the MG-GMRES method, respectively.

Table 1: Solution of Example 4.1 using the MG-GMRES method with parameters $\theta = \frac{\pi}{2}$ and $k = 3$.

N	Method	L_2^{Real}	L_2^{Imag}	$\lvert L_2\rvert$	$\lvert L_\infty\rvert$	COND
32	SFDM	8.55×10^{-5}	3.75×10^{-4}	3.85×10^{-4}	7.44×10^{-4}	1.229×10^{3}
	NSFDM	3.53×10^{-8}	1.54×10^{-7}	1.58×10^{-7}	3.06×10^{-7}	-
64	SFDM	2.20×10^{-5}	9.67×10^{-5}	9.91×10^{-5}	1.92×10^{-4}	4.779×10^{3}
	NSFDM	2.30×10^{-9}	8.29×10^{-9}	8.61×10^{-9}	1.73×10^{-8}	-
128	SFDM	5.60×10^{-6}	2.45×10^{-5}	2.52×10^{-5}	4.87×10^{-5}	1.883×10^{4}
	NSFDM	5.18×10^{-10}	6.40×10^{-9}	6.42×10^{-9}	1.57×10^{-8}	-
256	SFDM	1.41×10^{-6}	6.18×10^{-6}	6.34×10^{-6}	1.23×10^{-5}	7.477×10^{4}
	NSFDM	9.61×10^{-12}	4.21×10^{-11}	4.32×10^{-11}	8.35×10^{-11}	-
512	SFDM	3.54×10^{-7}	1.55×10^{-6}	1.59×10^{-6}	3.08×10^{-6}	2.979×10^{5}
	NSFDM	2.20×10^{-12}	9.40×10^{-12}	9.67×10^{-12}	1.87×10^{-11}	-

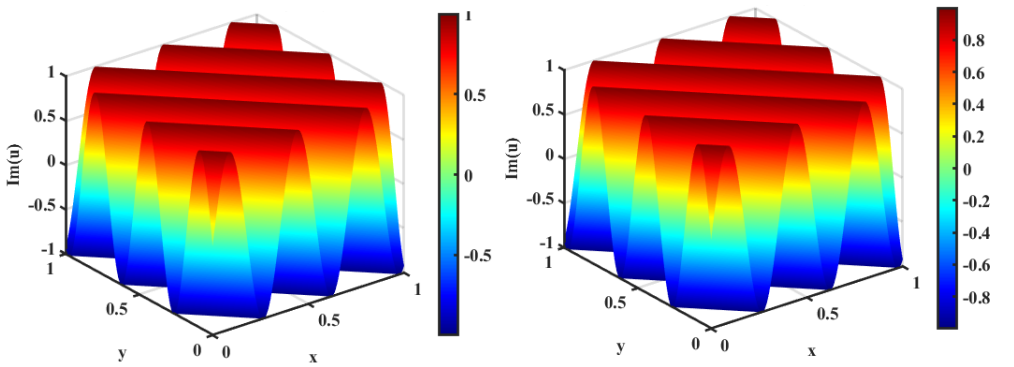
Table 2: Results for Example 4.1 obtained using the NSFD method with parameters $\theta = \frac{\pi}{4}$ and $k = 4$.

N	Method	L_2^{Real}	L_2^{Imag}	$\lvert L_2\rvert$	$\lvert L_\infty\rvert$	COND
32	BiCGSTAB	5.03×10^{-7}	1.73×10^{-7}	5.32×10^{-7}	1.07×10^{-6}	3.672×10^{3}
	MG	3.86×10^{-7}	6.15×10^{-8}	3.91×10^{-7}	7.85×10^{-7}	-
64	BiCGSTAB	3.19×10^{-8}	1.10×10^{-8}	3.37×10^{-8}	6.79×10^{-8}	1.429×10^{4}
	MG	2.94×10^{-8}	7.39×10^{-9}	3.04×10^{-8}	6.06×10^{-8}	-
128	BiCGSTAB	4.37×10^{-9}	1.35×10^{-9}	4.58×10^{-9}	8.98×10^{-9}	5.632×10^{4}
	MG	3.09×10^{-10}	3.01×10^{-10}	4.31×10^{-10}	9.69×10^{-10}	-
256	BiCGSTAB	9.77×10^{-10}	7.68×10^{-11}	9.80×10^{-10}	2.10×10^{-9}	2.236×10^{5}
	MG	1.30×10^{-10}	4.70×10^{-11}	1.38×10^{-10}	2.78×10^{-10}	-
512	BiCGSTAB	5.28×10^{-12}	1.70×10^{-12}	5.57×10^{-12}	1.24×10^{-11}	8.905×10^{5}
	MG	3.67×10^{-12}	2.30×10^{-12}	4.38×10^{-12}	9.01×10^{-12}	-

Table 3: Results for Example 4.1 obtained using the MG-GMRES method with parameters $\theta = \frac{\pi}{4}$ and $k = 12$.

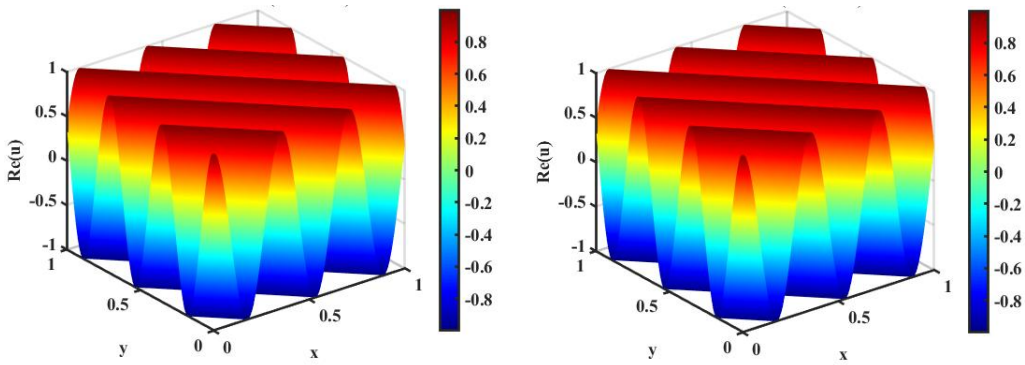
N	Method	$\lvert L_2\rvert$	L_∞^{Real}	L_∞^{Imag}	$\lvert L_\infty\rvert$	COND
32	SFDM	2.76×10^{-2}	7.40×10^{-2}	6.82×10^{-2}	1.00×10^{-1}	1.313×10^{3}
	NSFDM	9.37×10^{-5}	2.50×10^{-4}	2.28×10^{-4}	3.39×10^{-4}	-
64	SFDM	7.26×10^{-3}	1.94×10^{-2}	1.78×10^{-2}	2.63×10^{-2}	3.856×10^{3}
	NSFDM	6.23×10^{-6}	1.66×10^{-5}	1.52×10^{-5}	2.25×10^{-5}	-
128	SFDM	1.85×10^{-3}	4.96×10^{-3}	4.53×10^{-3}	6.72×10^{-3}	1.608×10^{4}
	NSFDM	4.47×10^{-7}	1.20×10^{-6}	1.07×10^{-6}	1.60×10^{-6}	-
256	SFDM	4.68×10^{-4}	1.25×10^{-3}	1.14×10^{-3}	1.69×10^{-3}	8.066×10^{4}
	NSFDM	2.54×10^{-7}	4.80×10^{-7}	3.56×10^{-7}	6.00×10^{-7}	-
512	SFDM	1.17×10^{-4}	3.14×10^{-4}	2.87×10^{-4}	4.25×10^{-4}	3.210×10^{5}
	NSFDM	7.00×10^{-8}	1.46×10^{-7}	8.30×10^{-8}	1.68×10^{-7}	-

For the case $k = 25$ and COND $= 8.279 \times 10^4$, the iterative solver BiCGSTAB exhibited divergence and failed to attain convergence within the prescribed tolerance and maximum number of iterations. This behavior indicates instability or inefficiency of the BiCGSTAB method for the given problem configuration at this parameter value. On the other hand, the multigrid method demonstrated robust performance, successfully converging to a highly accurate solution. The resulting error norms were notably small, with the discrete L_2-norm of the error reaching $|L_2| = 1.71 \times 10^{-6}$, and the maximum norm attaining $|L_\infty| = 7.2 \times 10^{-6}$. These results highlight the superior convergence properties and effectiveness of the MG method in handling the problem under consideration, particularly in contrast to BiCGSTAB at the same value of k. The results are illustrated in Figures 2–4.



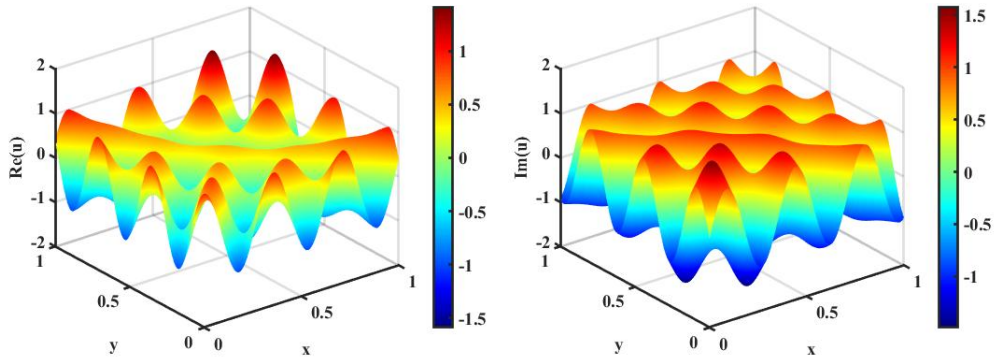
(a) (b)

Figure 2: (a) Approximated imaginary part of the solution for $N = 256$. (b) Analytical imaginary part, serving as a reference to validate the accuracy of the numerical solution.



(a) (b)

Figure 3: (a) Numerical approximation of the real part of the solution at $N = 256$. (b) Analytical real part used as a reference to validate the accuracy of the numerical results.



(a) (b)

Figure 4: (a) Numerical approximation of the real component of the solution at $N = 256$. (b) Numerical approximation of the imaginary component of the solution at $N = 256$ .Both figures using BiCGSTAB.

Example 4.2.

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + K^2 U = 0, \tag{14}$$

and the solution is given by,

$$U(x, y, z) = e^{i(k_1 x + k_2 y + k_3 z)}, \tag{15}$$

where

$$k_1 = k \cos\theta \cos\phi, \qquad k_2 = k \sin\theta \cos\phi, \qquad k_3 = k \sin\phi.$$

Tables 4 and 5 present the numerical results obtained using the MG-GMRES method and the NSFD method, respectively.

Table 4: Results for Example 4.2 obtained using the MG-GMRES method with parameters $k = \sqrt{2}$, $\theta = \frac{\pi}{8}$, and $\phi = 0$.

N	Method	L_2^{Real}	L_2^{Imag}	$\lvert L_2\rvert$	$\lvert L_\infty\rvert$
8	SFDM	6.180×10^{-5}	8.010×10^{-5}	1.015×10^{-4}	1.840×10^{-4}
	NSFDM	5.210×10^{-8}	6.740×10^{-8}	8.520×10^{-8}	1.520×10^{-7}
16	SFDM	1.770×10^{-5}	2.300×10^{-5}	2.900×10^{-5}	5.270×10^{-5}
	NSFDM	5.090×10^{-9}	6.630×10^{-9}	8.360×10^{-9}	1.120×10^{-8}
32	SFDM	4.710×10^{-6}	6.140×10^{-6}	7.740×10^{-6}	1.410×10^{-5}
	NSFDM	3.500×10^{-10}	4.580×10^{-10}	5.780×10^{-10}	1.060×10^{-9}
64	SFDM	1.220×10^{-6}	1.580×10^{-6}	2.000×10^{-6}	3.64×10^{-6}
	NSFDM	5.750×10^{-12}	6.950×10^{-12}	1.020×10^{-11}	3.000×10^{-11}

Table 5: Results for Example 4.2 obtained using the NSFD method with parameters $k = \sqrt{2}$, $\theta = \frac{\pi}{8}$, and $\phi = 0$.

N	Method	L_2^{Real}	L_2^{Imag}	$\lvert L_2\rvert$	$\lvert L_\infty\rvert$
8	BiCGSTAB	6.410×10^{-8}	8.290×10^{-8}	1.050×10^{-7}	1.960×10^{-7}
	MG	5.210×10^{-8}	6.740×10^{-8}	8.520×10^{-8}	1.520×10^{-7}
16	BiCGSTAB	1.420×10^{-8}	1.320×10^{-8}	1.940×10^{-8}	4.600×10^{-8}
	MG	5.090×10^{-9}	6.630×10^{-9}	8.360×10^{-9}	1.120×10^{-8}
32	BiCGSTAB	5.300×10^{-8}	8.650×10^{-8}	1.010×10^{-7}	2.230×10^{-7}
	MG	3.500×10^{-10}	4.580×10^{-10}	5.780×10^{-10}	1.060×10^{-9}
64	BiCGSTAB	2.144×10^{-7}	2.220×10^{-7}	3.090×10^{-7}	7.030×10^{-7}
	MG	5.750×10^{-12}	6.950×10^{-12}	1.020×10^{-11}	3.000×10^{-11}

5 Conclusion

This work presented a comparative study of iterative solvers for the ill-posed complex Helmholtz equation in 2D and 3D. The main novelty lies in integrating GMRES as a smoother within the Multigrid framework, which significantly improved robustness and convergence for indefinite and oscillatory systems. The results further demonstrated that the Multigrid method outperforms BiCGSTAB, particularly in highly oscillatory cases where BiCGSTAB often fails, while the use of NSFDM enhances stability and accuracy compared to the classical FDM. These findings highlight the originality and effectiveness of the proposed numerical framework for solving challenging Helmholtz problems.

For future work, this study can be extended in several directions. First, investigating other Krylov subspace methods and advanced preconditioning strategies may further improve convergence for highly indefinite Helmholtz systems. Second, applying the proposed Multigrid–GMRES framework to larger-scale and more realistic physical models, such as acoustics or electromagnetic scattering, would provide practical validation. Finally, extending the analysis to fractional or integral-equation formulations of the Helmholtz problem may offer additional insights into stability and efficiency.

Acknowledgement The authors are grateful to anonymous referees for their remarks and suggestions that helped improve the manuscript.

Conflicts of Interest The authors declare no conflict of interest.

References

[1] M. F. Adams (2004). Algebraic multigrid methods for constrained linear systems with applications to contact problems in solid mechanics. *Numerical Linear Algebra with Applications*, *11*(2-3), 141–153. https://doi.org/10.1002/nla.374.

[2] R. Anguelov, J. M. S. Lubuma & M. Shillor (2011). Topological dynamic consistency of non-standard finite difference schemes for dynamical systems. *Journal of Difference Equations and Applications*, *17*(12), 1769–1791. https://doi.org/10.1080/10236198.2010.488226.

[3] A. Bayliss, C. I. Goldstein & E. Turkel (1983). An iterative method for the Helmholtz equation. *Journal of Computational Physics*, *49*(3), 443–457. https://doi.org/10.1016/0021-9991(83)90139-0.

[4] W. L. Briggs, V. E. Henson & S. F. McCormick (2000). *A Multigrid Tutorial*. Other Titles in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, United States. https://doi.org/10.1137/1.9780898719505.

[5] E. Centofanti & S. Scacchi (2024). A comparison of algebraic multigrid bidomain solvers on hybrid CPU–GPU architectures. *Computer Methods in Applied Mechanics and Engineering*, *423*, Article ID: 116875. https://doi.org/10.1016/j.cma.2024.116875.

[6] H. Chen, P. Lu & X. Xu (2013). A hybridizable discontinuous Galerkin method for the Helmholtz equation with high wave number. *SIAM Journal on Numerical Analysis*, *51*(4), 2166–2188. https://doi.org/10.1137/120883451.

[7] Y. Chen, J. Cheng, T. Li & Y. Miao (2025). A learning based numerical method for Helmholtz equations with high frequency. *Journal of Computational Physics*, *520*, Article ID: 113478. https://doi.org/10.48550/arXiv.2401.09118.

[8] K. Christodoulou, O. Laghrouche, M. S. Mohamed & J. Trevelyan (2017). High-order finite elements for the solution of Helmholtz problems. *Computers & Structures*, *191*, 129–139. https://doi.org/10.1016/j.compstruc.2017.06.010.

[9] A. J. Cleary, R. D. Falgout, V. E. Henson, J. E. Jones, T. A. Manteuffel, S. F. McCormick, G. N. Miranda & J. W. Ruge (2000). Robustness and scalability of algebraic multigrid. *SIAM Journal on Scientific Computing*, *21*(5), 1886–1908. https://doi.org/10.1137/S1064827598339402.

[10] S. Cools, B. Reps & W. Vanroose (2014). A new level-dependent coarse grid correction scheme for indefinite Helmholtz problems. *Numerical Linear Algebra with Applications*, *21*(4), 513–533. https://doi.org/10.1002/nla.1895.

[11] Y. Dumont & J. M. S. Lubuma (2007). Non-standard finite difference schemes for multi-dimensional second-order systems in non-smooth mechanics. *Mathematical Methods in the Applied Sciences*, *30*(7), 789–825. https://doi.org/10.1002/mma.811.

[12] L. Eldén & V. Simoncini (2012). Solving ill-posed linear systems with GMRES and a singular preconditioner. *SIAM Journal on Matrix Analysis and Applications*, *33*(4), 1369–1394. https://doi.org/10.1137/110832793.

[13] H. C. Elman, O. G. Ernst & D. P. O'leary (2001). A multigrid method enhanced by Krylov subspace iteration for discrete Helmholtz equations. *SIAM Journal on Scientific Computing*, *23*(4), 1291–1315. https://doi.org/10.1137/S1064827501357190.

[14] Y. A. Erlangga (2008). Advances in iterative methods and preconditioners for the Helmholtz equation. *Archives of Computational Methods in Engineering*, *15*(1), 37–66. https://doi.org/10.1007/s11831-007-9013-7.

[15] Y. A. Erlangga, C. Vuik & C. W. Oosterlee (2004). On a class of preconditioners for solving the Helmholtz equation. *Applied Numerical Mathematics*, *50*(3-4), 409–425. https://doi.org/10.1016/j.apnum.2004.01.009.

[16] O. G. Ernst & M. J. Gander (2011). *Why it is Difficult to Solve Helmholtz Problems with Classical Iterative Methods*, volume 83 of *Lecture Notes in Computational Science and Engineering*, pp. 325–363. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-22061-6_10.

[17] A. Eseily, N. Sweilam, I. Hanafy & M. Ramadan (2025). Bi-conjugate gradient stabilized and multigrid approaches based on nonstandard finite differences for the solution of elliptic partial differential equations. *Alfarama Journal of Basic & Applied Sciences*, *6*(3), 304–320. https://doi.org/10.21608/ajbas.2025.385670.1255.

[18] F. Ghaffar, N. Badshah & S. Islam. Multigrid method for solution of 3D Helmholtz equation based on HOC schemes. *Abstract and Applied Analysis*, *2014*(1), Article ID: 954658. https://doi.org/10.1155/2014/954658.

[19] F. Ghaffar, N. Badshah, M. A. Khan & S. Islam (2014). Multigrid method for 2D Helmholtz equation using higher order finite difference scheme accelerated by Krylov subspace. *Journal of Applied Environmental and Biological Sciences*, *4*(10), 169–179. https://api.semanticscholar.org/CorpusID:124136722.

[20] W. Hackbusch (2013). *Multi-grid Methods and Applications* volume 4. Springer Science & Business Media, Berlin, Heidelberg. https://doi.org/10.1007/978-3-662-02427-0.

[21] A. Handlovičová & I. Riečanová (2016). Numerical solution to the complex 2D Helmholtz equation based on finite volume method with impedance boundary conditions. *Open Physics*, *14*(1), 436–443. https://doi.org/10.1515/phys-2016-0044.

[22] Z. He, P. Li, G. Zhao & H. Chen (2011). A meshless Galerkin least-square method for the Helmholtz equation. *Engineering Analysis with Boundary Elements*, *35*(6), 868–878. https://doi.org/10.1016/j.enganabound.2011.01.010.

[23] M. Heydari, M. Hosseininia & D. Baleanu (2023). An efficient method for 3D Helmholtz equation with complex solution. *AIMS Mathematics*, *8*(6), 14792–14819. https://doi.org/10.3934/math.2023756.

[24] R. J. LeVeque (2007). *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, United States. https://doi.org/10.1137/1.9780898717839.

[25] M. A. Londono & H. Montegranario (2019). Radial basis function-generated finite differences with Bessel weights for the 2D Helmholtz equation. *Engineering Analysis with Boundary Elements*, *105*, 257–266. https://doi.org/10.48550/arXiv.1812.04742.

[26] A. M. S. Mahdy (2025). Stability, existence, and uniqueness for solving fractional glioblastoma multiforme using a Caputo–Fabrizio derivative. *Mathematical Methods in the Applied Sciences*, *48*(7), 7360–7377. https://doi.org/10.1002/mma.9038.

[27] O. Z. Mehdizadeh & M. Paraschivoiu (2003). Investigation of a two-dimensional spectral element method for Helmholtz's equation. *Journal of Computational Physics*, *189*(1), 111–129. https://doi.org/10.1016/S0021-9991(03)00204-3.

[28] A. Nicholls & B. Honig (1991). A rapid finite difference algorithm, utilizing successive over-relaxation to solve the Poisson–Boltzmann equation. *Journal of Computational Chemistry*, *12*(4), 435–445. https://doi.org/10.1002/jcc.540120405.

[29] M. Pennacchio & V. Simoncini (2011). Fast structured AMG preconditioning for the bidomain model in electrocardiology. *SIAM Journal on Scientific Computing*, *33*(2), 721–745. https://doi.org/10.1137/100796364.

[30] G. Plank, M. Liebmann, R. W. dos Santos, E. J. Vigmond & G. Haase (2007). Algebraic multigrid preconditioner for the cardiac bidomain model. *IEEE Transactions on Biomedical Engineering*, *54*(4), 585–596. https://doi.org/10.1109/TBME.2006.889181.

[31] N. Raza, A. Bakar, A. Khan, C. Tunç et al. (2022). Numerical simulations of the fractional-order SIQ mathematical model of corona virus disease using the nonstandard finite difference scheme. *Malaysian Journal of Mathematical Sciences*, *16*(3), 391–411. https://doi.org/10.47836/mjms.16.3.01.

[32] L. Reichel & Q. Ye (2005). Breakdown-free GMRES for singular systems. *SIAM Journal on Matrix Analysis and Applications*, *26*(4), 1001–1021. https://doi.org/10.1137/S0895479803437803.

[33] K. Sayevand, J. T. Machado & V. Moradi (2019). A new non-standard finite difference method for analyzing the fractional Navier–Stokes equations. *Computers & Mathematics with Applications*, *78*(5), 1681–1694. https://doi.org/10.1016/j.camwa.2018.12.016.

[34] G. D. Smith (1985). *Numerical Solution of Partial Differential Equations: Finite Difference Methods*. Oxford University Press, New York, United States.

[35] N. Song & E. Lee (2021). Dual system least squares finite element method for the Helmholtz equation. *Results in Applied Mathematics*, *9*, Article ID: 100138. https://doi.org/10.1016/j.rinam.2020.100138.

[36] P. Sonneveld (1989). CGS, a fast Lanczos-type solver for nonsymmetric linear systems. *SIAM Journal on Scientific and Statistical Computing*, *10*(1), 36–52. https://doi.org/10.1137/0910004.

[37] U. Trottenberg, C. W. Oosterlee & A. Schuller (2001). *Multigrid*. Academic Press, San Diego, California.

[38] H. A. Van der Vorst (1992). Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of nonsymmetric linear systems. *SIAM Journal on scientific and Statistical Computing*, *13*(2), 631–644. https://doi.org/10.1137/0913035.

[39] H. A. Van der Vorst (2003). *Iterative Krylov Methods for Large Linear Systems*. Cambridge University Press, Cambridge, United Kingdom. https://doi.org/10.1017/CBO9780511615115.

[40] W. Vanroose, B. Reps & H. B. Zubair (2011). Multigrid preconditioners based on polynomial smoothers for the Helmholtz equation with absorbing layers. In *International Conference on Preconditioning Techniques for Scientific and Industrial Applications* (*Preconditioning 2011*), pp. 1–3. Bordeaux, France.

[41] H. Wendland (2017). *Numerical Linear Algebra: An Introduction* volume 56. Cambridge University Press, Selangor, Malaysia. https://doi.org/10.1017/9781316544938.

[42] M. M. Xu, J. Sulaiman & L. H. Ali (2022). Refinement of SOR iterative method for the linear rational finite difference solution of second-order Fredholm integro-differential equations. *Malaysian Journal of Mathematical Sciences*, *16*(1), 105–117. https://doi.org/10.47836/mjms.16.1.09.

[43] Q. Zou (2023). GMRES algorithms over 35 years. *Applied Mathematics and Computation*, *445*, Article ID: 127869. https://doi.org/10.48550/arXiv.2110.04017.